

On the factorization of the polar of a plane branch

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Abstract

In this paper we present the most complete description as possible of the factorization of the general polar of the general member of an equisingularity class of irreducible germs of complex plane curves. Our result will refine the rough description of the factorization given by M. Merle in [M] and it is based on the result given by E. Casas-Alvero in [C2] that describes the cluster of the singularities of such polars. By using our analysis, it will be possible to characterize all equisingularity classes of irreducible plane germs with r characteristic exponents having the exceptional behavior that the general polar of a general curve in this equisingularity class has only irreducible components with less than r characteristic exponents, generalizing a result obtained for $r = 2$ in [HHI1].

1 Introduction

The study of the polar of a germ of plane curve is a classical subject and the topological or equisingular classification of polars of equisingular complex plane curve germs is still an open problem. These objects have been used in classical algebraic geometry for enumeration purposes, such as Plücker formulas, and were resuscitated during the 70's in the work of B. Teissier [T] for the study of families of singular hypersurfaces, being still actively studied nowadays.

Initially, it was thought that topologically equivalent germs of plane curves had topologically equivalent polar curves, which is false as shown with a simple example in [P]. The topological type of the general polar of the germ of a plane curve is actually an analytic invariant of the germ. However,

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there are some particular invariants attached to the polars of topologically equivalent plane curve germs, namely the polar quotients, associated to the decomposition of the polar, roughly described by a Theorem of Merle in [M], as we will explicit later. Actually, in this paper we will describe topologically the complete decomposition of the general polar of the general complex plane curve germ topologically equivalent to a given irreducible one. This will be done by using the description given by E. Casas-Alvero in [C2] of the cluster of the general polar of the general curve belonging to an equisingularity class of irreducible germs of complex plane curves. We explore this cluster in order to recover the characteristic exponents of each irreducible component of the general polar and the intersection multiplicities of all pairs of such components. This determines in Zariski's way the equisingularity class of the general polar of the general member of the equisingularity class of the curve. Our analysis will allow us, as a byproduct, to characterize the equisingularity classes of irreducible plane germs such that their general members have general polar that admit only irreducible components with at most one less characteristic exponent, generalizing a result obtained in [HHI1] in the case of curves with two characteristic exponents.

2 Classical results

A germ of an analytic plane curve at the origin of \mathbb{C}^2 is a germ of set $C = C_f = \{(x, y) \in (\mathbb{C}^2, 0); f(x, y) = 0\}$, where $f \in \mathbb{C}\{x, y\}$ is a convergent complex power series in two variables at the origin. Two such germs will be considered analytically equivalent if there is a germ of analytic diffeomorphism φ of $(\mathbb{C}^2, 0)$, also called an analytic change of coordinates, such that $\varphi(C_f) = C_g$. When the above φ is just a homeomorphism, we say that C_f and C_g are topologically equivalent, or equisingular, writing, in this case, $C_f \equiv C_g$.

From now on, we will assume that f is an irreducible power series and call its associated curve $C = C_f$ a branch. After an analytic change of coordinates, if necessary, we may assume that $I(f, x) = n < m = I(f, y)$ and $n \nmid m$, where $I(f, g)$ stands for the intersection multiplicity at the origin of the plane curve germs C_f and C_g . The integer n is called the multiplicity of C . With such coordinates suitably chosen, it is well known that a branch C admits a Newton-Puiseux parametrization of the form (t^2, t^m) , if $n = 2$, or $(t^n, \sum_{m \leq i < c} a_i t^i)$, if $n > 2$, where c is some positive integer, called the conductor of C . Conversely, given such a parametrization, attached to it there is a well defined branch. It is also classically known that the

topological, or equisingularity class of C is completely determined by n and the characteristic exponents m_1, \dots, m_r , defined by

$$m_i = \min\{j; a_j \neq 0 \text{ and } e_{i-1} \nmid j\},$$

where $e_0 = n$ and, for $k > 0$, $e_k = \gcd(n, m_1, \dots, m_k)$ and $e_r = 1$. The integer r is what we call the genus of C . We also define the integers $d_0 = 1$ and $d_i = \frac{e_{i-1}}{e_i}$, for $i = 1, \dots, r$.

When a germ of curve is not irreducible, but reduced, Zariski has shown that its equisingularity type is determined by the equisingularity type of its branches and by their mutual intersection multiplicities.

In what follows, we will consider the set $K(n, m_1, \dots, m_r)$ that parametrizes all Newton-Puiseux finite expansions as above with multiplicity n and characteristic exponents m_1, \dots, m_r .

Let f be a reduced power series. The germ of curve defined by $P_{(a:b)}(f) := af_x + bf_y = 0$ is the polar curve of f in the direction $(a : b) \in \mathbb{P}^1$. When $(a : b)$ is a general point of \mathbb{P}^1 , we say that the associated polar $P_{(a:b)}(f) = 0$ is general and we denote it shortly by $P(f)$.

In this paper we only consider the general polar of f and we refer to it simply as the polar curve.

In general, the polar curve depends upon the equation f of C_f , however its topological type depends only upon the analytic type of C_f (see [C3, Theorem 7.2.10]).

The next result due to M. Merle provides a rough decomposition of $P(f)$ in packages of curves, not necessarily irreducible, that gives partial information about the topology of $P(f)$.

Theorem 2.1 (Merle [M]). *Let C_f be a germ of an irreducible curve with multiplicity n and characteristic exponents m_1, \dots, m_r . Then the general polar $P(f)$ has a decomposition of the form*

$$P(f) = \gamma_1 \gamma_2 \cdots \gamma_r,$$

where each γ_i , not necessarily irreducible, satisfies the following conditions:

- i) The multiplicity of γ_i is given by $m(\gamma_i) = d_0 d_1 d_2 \cdots d_{i-1} (d_i - 1)$;
- ii) Each irreducible factor $\gamma_{i,j}$ of γ_i satisfies

$$\frac{I(\gamma_{i,j}, f)}{m(\gamma_{i,j})} = \frac{1}{n} \sum_{k=1}^{i-1} (e_{k-1} - e_k) m_k + m_i.$$

Let us make some few remarks. Merle's Theorem does not describe completely the topology of $P(f)$, because it does not describe the branches inside each package γ_i . Such branches depend upon the analytic type of f and not only upon its topological type. It also does not describe the intersection multiplicities among the branches of the polar. The terms in the second conclusion are the so called polar quotients and the equality says that the branches $\gamma_{i,j}$ have contact order with C_f equal to m_i , which implies that they have genus at least $i - 1$, but they may have greater genus.

On the other hand, Casas in [C2], determines the equisingularity class of $P(f)$, for an f corresponding to a general member of $K(n, m_1, \dots, m_r)$ in terms of a certain weighted cluster obtained from the Enriques diagram attached to the resolution of C_f .

If $r = 1$, Casas in [C1] describes more explicitly the factorization of $P(f)$ as follows:

Let n and m be two coprime natural numbers. Consider the euclidean GCD algorithm applied to the pair n, m :

$$\begin{aligned} m &= h_0 n + n_1 \\ n &= h_1 n_1 + n_2 \\ n_1 &= h_2 n_2 + n_3 \\ &\vdots \\ n_{s-2} &= h_{s-1} n_{s-1} + 1 \\ n_{s-1} &= h_s 1. \end{aligned}$$

We denote by $\frac{m}{n} = [h_0, \dots, h_s]$ the partial fraction decomposition of $\frac{m}{n}$, adjusted in such a way that s becomes even, say $s = 2t$ (for example, $[a_0, a_1] = [a_0, a_1 - 1, 1]$). Put $\frac{q_i}{p_i} = [h_0, \dots, h_i]$ in such a way that q_i and p_i are coprime. So, one has the following theorem:

Theorem 2.2 (Casas-Alvero [C1]). *If f is a general member of $K(n, m)$ where $\gcd(n, m) = 1$, then $P(f)$ has one Merle package with branches $\gamma_{i,j}$, $i = 1, \dots, t$, $j = 1, \dots, h_{2i}$, having multiplicity $I(f, X) = p_{2i-1}$ and $I(f, Y) = q_{2i-1}$ and such that*

$$I(\gamma_{i,j}, \gamma_{i',j'}) = \min(p_{2i-1} q_{2i'-1}, p_{2i'-1} q_{2i-1}).$$

Remark 2.3. *Notice that the branches of $P(f)$ for a general $f \in K(n, m)$ are all smooth if and only if $p_{2i-1} = 1$, for all i . But, since the p_i form an increasing sequence, this only may happen when $2t - 1 = 1$, that is, $t = 1$.*

If $\frac{m}{n} = [h_0, h_1, h_2]$, then we have $m = h_0 n + n_1$; $n = h_1 n_1 + 1$; $n_1 = h_2 \cdot 1$. The condition that $\frac{q_1}{p_1} = [h_0, h_1]$ is an integer is equivalent to $h_1 = 1$ and

$h_2 = n - 1$. Hence the fact that $P(f)$ has only smooth branches is equivalent to $m = (h_0 + 1)n - 1$.

In the case where $\frac{m}{n} = [h_0, h_1 - 1, 1]$, so $\frac{q_1}{p_1} = [h_0, h_1 - 1]$. Now, the condition that $\frac{q_1}{p_1}$ is an integer is equivalent to $h_1 = 2$ and this in turn is equivalent to $n = 2$. Hence, the fact $P(f)$ has only smooth branches is equivalent to $m = h_0 \cdot 2 + 1 = (h_0 + 1)2 - 1$.

In conclusion, one has that $P(f)$, where f corresponds to a general member of $K(n, m)$, has only smooth branches, if and only if $m = \lambda n - 1$, where λ is some natural number greater than 1.

2.1 The infinitely near points

Let $S_0 \subset \mathbb{C}^2$ be an open set containing the origin $0 = (0, 0)$. Let $\pi: S_1 \rightarrow S_0$ be the blow-up of S_0 centered at 0 and denote by $E_0 = \pi^{-1}(0)$ the exceptional divisor of π . We denote by \mathcal{N}_0 the set of infinitely near points to 0, which can be viewed as the disjoint union of 0 and all exceptional divisors obtained by successive blowing-ups above 0. The set of points on the exceptional divisor of the i -th blow-up centered at a point $P \in S_{i-1}$ are called the first infinitesimal neighborhood of P and the i -th infinitesimal neighborhood of 0. The set \mathcal{N}_0 is naturally endowed with an order relation defined by $P < Q$ if and only if $Q \in \mathcal{N}_P$.

Given $f \in \mathbb{C}\{x, y\}$ that defines a curve C and given P in the first infinitesimal neighborhood of 0, we denote by C_P the germ of curve at P defined via the strict transform \tilde{f}_P of f , which might be viewed as the germ at P of the closure of $\pi^{-1}(C \setminus \{0\})$. By induction we may obtain the strict transform of C at any point of \mathcal{N}_0 .

The multiplicity of C_P at $P \in \mathcal{N}_0$ is $m_P(f) = m_P(\tilde{f}_P)$. We say that P lies on C , or belongs to it, if and only if $m_P(f) > 0$, and denote by $\mathcal{N}_0(f)$ the set of all such points. A point $P \in \mathcal{N}_0(f)$ is *simple* (resp. *multiple*) if and only if $m_P(f) = 1$ (resp. $m_P(f) > 1$). Given two germs of curves C_f and C_g , their intersection multiplicity at 0 can be computed by means of Noether's formula as follows:

$$I(f, g) = \sum_{P \in \mathcal{N}_0(f) \cap \mathcal{N}_0(g)} m_P(f) m_P(g). \quad (1)$$

Given $P, Q \in \mathcal{N}_0$ such that $P < Q$, we say that Q is *proximate* to P (writing $Q \rightarrow P$) if and only if Q lies on the exceptional divisor E_P or in the strict transform of E_P . A point P is said to be *free* (resp. *satellite*) if it is proximate to exactly one point (resp. two points), and these are the only possibilities. Notice that $Q \rightarrow P$ implies $Q > P$, but not conversely.

An important formula due to Noether is the following:

$$m_P(f) = \sum_{Q \rightarrow P} m_Q(f).$$

A point $P \in \mathcal{N}_0(f)$ is *singular* if it is either multiple, or satellite, or precedes a satellite point on C_f , and it is *non-singular*, or *regular*, otherwise. Equivalently, P is non-singular if and only if it is free and there is no satellite point $Q > P$.

Let $C_f = \bigcup_{i=1}^s C_{f_i}$ be a reducible plane curve we denote by $P_i \in \mathcal{N}_0(f)$ the first regular point on C_{f_i} . We denote by

$$S(f) = \{Q \in \mathcal{N}_0(f); Q = P_i \text{ or } Q \text{ is singular}\}.$$

It may be shown that two curves C_f and C_g are equisingular if and only if there exists a bijection $\phi: S(f) \rightarrow S(g)$ such that both ϕ, ϕ^{-1} preserve the natural ordering and the proximity relations among their infinitely near points.

Definition 2.4. A cluster \mathcal{K} is a finite subset $K \subset \mathcal{N}_0$ such that if $P \in K$, then any other point $Q < P$ also belongs to K , together with a valuation $v_{\mathcal{K}}: K \rightarrow \mathbb{Z}$. The set K is called the support of \mathcal{K} and the number $v_{\mathcal{K}}(P)$ is the virtual multiplicity of P in \mathcal{K} .

We follow Casas, representing a cluster by means of an Enriques diagram, which is a tree whose vertices are identified with the points in K (the root corresponds to the origin 0) and there is an edge between P and Q if and only if P lies on the first neighborhood of Q or vice-versa. Moreover, the edges are drawn according to the following rules:

- i) If Q is free and proximate to P , the edge joining P and Q is curved and if $P \neq 0$, it is tangent to the edge ending at P .
- ii) If P and Q (Q in the first neighborhood of P) have been represented, the other points proximate to P in successive neighborhoods of Q are represented on a straight half-line starting at Q and orthogonal to the edge ending at Q .

Definition 2.5. We will say that a curve C_f goes sharply through the cluster \mathcal{K} if C_f goes through K with effective multiplicities equal to the virtual ones and has no singular points outside of K .

2.2 Enriques' Theorem

In what follows we will describe the *cluster of singularities* of a plane branch C_f , that is, the cluster $\mathcal{K}(f) = (S(f), v_{\mathcal{K}(f)})$, where $v_{\mathcal{K}(f)}(P) = m_P(f)$.

Suppose that C_f has multiplicity n and characteristic exponents m_1, \dots, m_r , then C_f is analytically equivalent to a curve that admits a Puiseux parametrization of the form $x = t^n$, $y = \sum_{i \geq m_1} a_i t^i$ such that $a_{m_k} \neq 0$ for $k = 1, \dots, r$ and $a_{m_1} = 1$.

Denoting $m_0 = 0$, $n_0^k = e_{k-1} = \gcd(n, m_1, \dots, m_{k-1})$, $n_0^{k+1} = n_{s(k)}^k = e_k$, we consider the euclidean expansions

$$\begin{aligned} m_k - m_{k-1} &= h_0^k n_0^k + n_1^k \\ n_0^k &= h_1^k n_1^k + n_2^k \\ n_1^k &= h_2^k n_2^k + n_3^k \\ &\vdots \\ n_{s(k)-2}^k &= h_{s(k)-1}^k n_{s(k)-1}^k + n_{s(k)}^k \\ n_{s(k)-1}^k &= h_{s(k)}^k n_{s(k)}^k. \end{aligned}$$

When $k = 1$, we omit the index k in n_j^k , h_j^k and $s(k)$.

The cluster of C is composed by r blocks, which we describe below.

The first block is composed as follows:

It starts with the point $P_{0,1} = O$, followed by points $P_{0,i} \in \mathcal{N}_0(f)$, $i = 2, \dots, h_0$, each one in the first neighborhood of the preceeding one, all free with value n .

It continues with the point $P_{1,1}$, free in the first neighborhood of P_{0,h_0} , followed by points $P_{1,i}$, $i = 2, \dots, h_1$, not free and each in the first neighborhood of the preceeding one, with value n_1 .

For $2 \leq j \leq s$, the point $P_{j,1}$ is proximate to $P_{j-2,h_{j-2}}$ and for $i = 2, \dots, h_j$ we have $P_{j,i}$ proximate to $P_{j-1,h_{j-1}}$ in the first neighborhood of $P_{j,i-1}$ with value n_j .

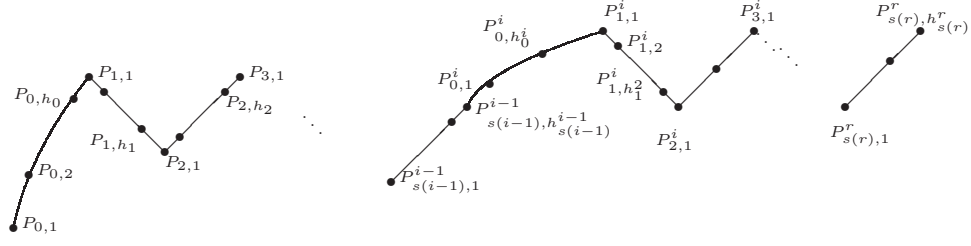
For $1 < k \leq r$, we put $P_{s(k-1),h_{s(k-1)}}^{k-1} = P_{0,0}^k$. The points of the cluster in the k -th block after $P_{0,0}^k$ are given by:

h_0^k free points $P_{0,1}^k, \dots, P_{0,h_0^k}^k$ with value n_0^k ;

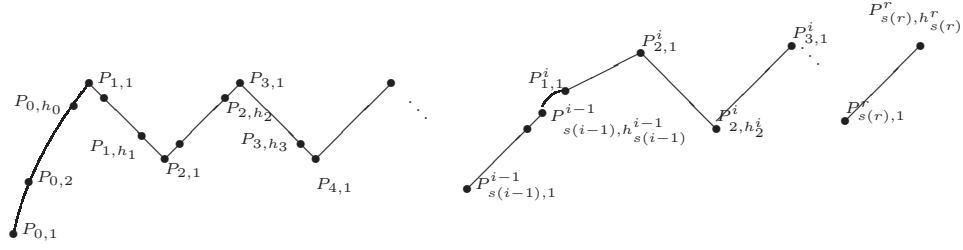
h_1^k points $P_{1,1}^k, \dots, P_{1,h_1^k}^k$ with value n_1^k proximate to $P_{0,h_0^k}^k$.

For $2 \leq j \leq s(k)$, we have h_j^k points $P_{j,1}^k, \dots, P_{j,h_j^k}^k$, where the first one is proximate to $P_{j-2,h_{j-2}^k}^k$ and for $i = 2, \dots, h_j^k$, the point $P_{j,i}^k$ is proximate to $P_{j-1,h_{j-1}^k}^k$ and all of them have value n_j^k .

This yields to the following Enriques diagrams:



if $m_i - m_{i-1} > e_{i-1}$.



if $m_i - m_{i-1} < e_{i-1}$.

3 Description of the packages in Merle's Theorem

By Casas' theorem [C2], we have that the cluster \mathcal{K}^r of the polar of a branch corresponding to a general member of $K(n, m_1, \dots, m_r)$ has the same support K^r as the cluster $\mathcal{K}(f)$ of the singularities of C_f , that is

$$K^r = \{P_{i,j}^k; 1 \leq k \leq r, 0 \leq i \leq s(k), 1 \leq j \leq h_{s(i)}^k\},$$

with valuation:

$$v_{\mathcal{K}^r}(P_{i,j}^k) = \begin{cases} m_{P_{s(k),h_{s(k)}^k}^k}(f) - 1, & \text{if } (i,j) = (s(k), h_{s(k)}^k); \text{ otherwise,} \\ m_{P_{i,j}^k}(f) - 1, & \text{if } i \text{ is even,} \\ m_{P_{i,j}^k}(f), & \text{if } i \text{ is odd.} \end{cases} \quad (2)$$

To describe explicitly Merle's packages of such a polar, we firstly consider the cluster \mathcal{K}' given as follows:

1. If $m_r - m_{r-1} > e_{r-1}$, then its support is $K' = \{P_{i,j}^r; 0 \leq i \leq s(r), 1 \leq j \leq h_{s(r)}^r\}$, with valuation:

$$v_{\mathcal{K}'}(P_{i,j}^r) = \begin{cases} 0, & \text{if } (i,j) = (s(r), h_{s(r)}^r); \text{ otherwise,} \\ n_i^r - 1, & \text{if } i \text{ is even,} \\ n_i^r, & \text{if } i \text{ is odd.} \end{cases}$$

Notice that \mathcal{K}' represents the cluster of the polar of a general curve f_r in $K(e_{r-1}, m_r - m_{r-1})$ based at $P_{1,0}^r$.

2. If $m_r - m_{r-1} < e_{r-1}$, then its support is

$$K' = \{P_{i,j}^r; 1 \leq i \leq s(r), 1 \leq j \leq h_{s(r)}^r\} \cup \{P_{s(r-1), h_{s(r-1)}^{r-1}}^{r-1}\},$$

with same values as above on the first set and

$$v_{\mathcal{K}'}(P_{s(r-1), h_{s(r-1)}^{r-1}}^{r-1}) = e_{r-1} - 1.$$

Notice that this represents the cluster of the polar of a general curve f_r in $K(e_{r-1}, m_r - m_{r-1} + e_{r-1})$ based at $P_{s(r-1), h_{s(r-1)}^{r-1}}^{r-1}$.

Now, by Theorem 2.2, we have that:

$$P(f_r) = \prod_{i=1}^{\lfloor \frac{s(r)+1}{2} \rfloor} \prod_{j=1}^{h_{2i}^r} \gamma_{i,j}^r$$

where $\gamma_{i,j}^r$ is determined by $m_r - m_{r-1}$ and e_{r-1} according to the following cases:

- 1'. If $m_r - m_{r-1} > e_{r-1}$, writing $\frac{m_r - m_{r-1}}{e_{r-1}} = [h_0^r, \dots, h_{s(r)}^r]$, one has that $\gamma_{i,j}^r \in K(p_{2i-1}^r, q_{2i-1}^r)$, where $\frac{q_{2i-1}^r}{p_{2i-1}^r} = [h_0^r, \dots, h_{2i-1}^r]$ and $\gcd(p_{2i-1}^r, q_{2i-1}^r) = 1$.
- 2'. If $m_r - m_{r-1} < e_{r-1}$, writing $\frac{m_r - m_{r-1} + e_{r-1}}{e_{r-1}} = [1, h_1^r, \dots, h_{s(r)}^r]$, one has that $\gamma_{i,j}^r \in K(p_{2i-1}^r, p_{2i-1}^r + q_{2i-1}^r)$, where $\frac{q_{2i-1}^r}{p_{2i-1}^r} = [0, h_1^r, \dots, h_{2i-1}^r]$.

Now, by blowing down the branches $\gamma_{i,j}^r$ to the point $P_{0,1}$, with respect to the cluster of singularities of any element in $K(\tilde{n}, \tilde{m}_1, \dots, \tilde{m}_{r-1})$, where $\tilde{n} = n/e_{r-1}$ and $\tilde{m}_i = m_i/e_{r-1}$, $i = 1, \dots, r-1$, we get branches $\xi_{i,j}^r$ that

pass through the points $K^{r-1} \cup K'_i$, where $K^{r-1} = \{P_{i,j}^k; 1 \leq k \leq r-1, 0 \leq i \leq s(k), 1 \leq j \leq h_{s(i)}^i\}$, and $K'_i = \{P_{0,1}^r, \dots, P_{2i-1, h_{2i-1}^r}\}$, with multiplicities at the points of K^{r-1} given by

$$m_{P_{i,j}^k}(\xi_{i,j}^r) = \frac{m_{P_{i,j}^k}(f)}{e_{r-1}} p_{2i-1}^r, \quad k = 1, \dots, r-1;$$

and the multiplicities at the points of K' given according the following cases:

1''. For $m_r - m_{r-1} > e_{r-1}$ we have that the multiplicities of the $\xi_{i,j}^r$ at the points $P_{0,1}^r, \dots, P_{2i-1, h_{2i-1}^r}$ are determined by $\frac{q_{2i-1}^r}{p_{2i-1}^r}$, then by definition of $\xi_{i,j}^r$ we have that the strict transform of the curve ξ_r in the point $P_{0,1}^r$ goes sharply through the cluster \mathcal{K}' , since the strict transform of $\xi_{i,j}^r$ at the point $P_{0,1}^r$ coincides with $\gamma_{i,j}^r$.

2''. For $m_r - m_{r-1} < e_{r-1}$ we have that the multiplicities of the $\xi_{i,j}^r$ at the points $P_{s(r-1), h_{s(r-1)}^{r-1}}, P_{0,1}^r, \dots, P_{2i-1, h_{2i-1}^r}$ are determined by $1 + \frac{q_{2i-1}^r}{p_{2i-1}^r}$ and, from the definition of $\xi_{i,j}^r$, the strict transform of curve ξ_r at the point $P_{s(r-1), h_{s(r-1)}^{r-1}}^{r-1}$ goes sharply through the cluster \mathcal{K}' , for the same reason as above.

From the above analysis, one sees that

$$\xi_r = \prod_{i=1}^{\lfloor \frac{s(r)+1}{2} \rfloor} \prod_{j=1}^{h_{2i}^r} \xi_{i,j}^r,$$

with

$$\xi_{i,j}^r \in K(p_{2i-1}^r \tilde{n}, p_{2i-1}^r \tilde{m}_1, \dots, p_{2i-1}^r \tilde{m}_{r-1}, p_{2i-1}^r \tilde{m}_{r-1} + q_{2i-1}^r).$$

In order to describe the decomposition of the polar of f we consider the cluster $\overline{\mathcal{K}}$ whose support is the same as that of $\mathcal{K}(f)$ (or of \mathcal{K}^r), with valuation $v_{\overline{\mathcal{K}}}(P_{i,j}^k) = v_{\mathcal{K}^r}(P_{i,j}^k) - m_{P_{i,j}^k}(\xi_r)$.

In particular, we have that $v_{\overline{\mathcal{K}}}(P_{i,j}^r) = 0$ and if $m_r - m_{r-1} < e_{r-1}$, then

$$v_{\overline{\mathcal{K}}}(P_{r-1, s(r-1)}^{r-1}) = (e_{r-1} - 1) - (e_{r-1} - 1) = 0.$$

By a computation, using the proximity relations, one obtains

$$v_{\overline{\mathcal{K}}}(P_{i,j}^{r-1}) = \begin{cases} (\tilde{n}_i^{r-1} e_{r-1} - 1) - (e_{r-1} - 1) \tilde{n}_i^{r-1} = \tilde{n}_i^{r-1} - 1, & \text{if } i \text{ is even,} \\ \tilde{n}_i^{r-1} e_{r-1} - (e_{r-1} - 1) \tilde{n}_i^{r-1} = \tilde{n}_i^{r-1}, & \text{if } i \text{ is odd.} \end{cases}$$

where $\tilde{n}_i^{r-1} = \frac{n_i^{r-1}}{e_{r-1}}$.

Using Noether's formulas and by a similar argument, it is possible to show that $v_{\overline{\mathcal{K}}}(P_{i,j}^k) = \tilde{n}_i^k - 1$, if i is even, and $v_{\overline{\mathcal{K}}}(P_{i,j}^k) = \tilde{n}_i^k$, if i is odd.

Finally, in any situation we have $v_{\overline{\mathcal{K}}}(P_{s(k),h_{s(k)}^k}^k) = \tilde{n}_{s(k)}^k - 1$. In this way, the cluster $\overline{\mathcal{K}}$ represents the cluster of singularities of the polar curve of a generic branch g in $K(\tilde{n}, \tilde{m}_1, \dots, \tilde{m}_{r-1})$. Therefore,

$$P(f) = P(g)\xi_r.$$

Now, repeating the same procedure to $P(g)$, and so on, we obtain

$$P(f) = \xi_1 \cdots \xi_{r-1} \xi_r,$$

where $\xi_1 = P(f_1)$ and f_1 is a general member of $K(\frac{n}{e_1}, \frac{m_1}{e_1})$, which is explicitly described in Theorem 2.2. On the other hand,

$$\xi_{k+1} = \prod_{i=1}^{\lfloor \frac{s(k+1)+1}{2} \rfloor} \prod_{j=1}^{h_{2i}} \xi_{i,j}^{k+1}, \quad k = 1, \dots, r-1, \quad (3)$$

where, if we write $\frac{m_{k+1}-m_k}{e_k} = [h_0^{k+1}, \dots, h_{s(k+1)}^{k+1}]$ and define

$$\frac{q_{2i-1}^{k+1}}{p_{2i-1}^{k+1}} = [h_0^{k+1}, h_1^{k+1}, \dots, h_{2i-1}^{k+1}], \quad \text{with } \gcd(p_{2i-1}^{k+1}, q_{2i-1}^{k+1}) = 1,$$

we have

$$\xi_{i,j}^{k+1} \in K(p_{2i-1}^{k+1} \frac{n}{e_k}, p_{2i-1}^{k+1} \frac{m_1}{e_k}, \dots, p_{2i-1}^{k+1} \frac{m_k}{e_k}, p_{2i-1}^{k+1} \frac{m_k}{e_k} + q_{2i-1}^{k+1}).$$

Summarizing, we have proved part of the following result.

Theorem 3.1. *If f is a general branch in $K(n, m_1, \dots, m_r)$, then the Merle decomposition of $P(f)$ is given by*

$$P(f) = \xi_1 \xi_2 \cdots \xi_r,$$

where $\xi_1 = P(f_1)$ with f_1 a general member of $K(\frac{n}{e_1}, \frac{m_1}{e_1})$ and ξ_{k+1} is as in (3).

The intersection multiplicities of these branches are given by:

$$I(\xi_{i,j}^{k+1}, \xi_{u,v}^{k+1}) = p_{2i-1}^{k+1} p_{2u-1}^{k+1} \left(\frac{n}{e_k^2} + \sum_{w=1}^{k-1} \frac{e_w}{e_k^2} (m_{w+1} - m_w) \right) + q_{2u-1}^{k+1} p_{2i-1}^{k+1}, \quad \text{for } i \leq u.$$

$$I(\xi_{i,j}^{l+1}, \xi_{u,v}^{k+1}) = \frac{p_{2i-1}^{l+1} p_{2u-1}^{k+1}}{e_l e_k} \left(\sum_{w=1}^l m_w (e_{w-1} - e_w) + m_{l+1} e_l \right), \quad \text{for } k > l \geq 0,$$

with the convention that $\sum_{w=s}^t A_w = 0$, if $t < s$.

Proof: It remains only to compute the intersection multiplicities.

By Noether's formula, we know that the intersection multiplicity of two branches is the sum of the products of the multiplicities in common points.

Case 1. The branches belong to the same package.

Suppose that $1 \leq i \leq u \leq \left\lceil \frac{s(k+1)+1}{2} \right\rceil$, $1 \leq j \leq h_{2i}$ and $1 \leq v \leq h_{2u}$ and let

$$\xi_{i,j}^{k+1} \in K(p_{2i-1}^{k+1} \frac{n}{e_k}, p_{2i-1}^{k+1} \frac{m_1}{e_k}, \dots, p_{2i-1}^{k+1} \frac{m_i}{e_k}, p_{2i-1}^{k+1} \frac{m_i}{e_k} + q_{2i-1}^{k+1}), \text{ and}$$

$$\xi_{u,v}^{k+1} \in K(p_{2u-1}^{k+1} \frac{n}{e_k}, p_{2u-1}^{k+1} \frac{m_1}{e_k}, \dots, p_{2u-1}^{k+1} \frac{m_i}{e_k}, p_{2u-1}^{k+1} \frac{m_i}{e_k} + q_{2u-1}^{k+1}).$$

As $i \leq u$, we have that the last common point of the two above branches is $P_{2i-1, h_{2i-1}}^{k+1}$. Using the clusters of both branches, we obtain that the sum of products of the multiplicities until the point $P_{s(k), h_{s(k)}}^k$ is

$$\left(\frac{e_1}{e_k^2} m_1 + \sum_{j=1}^{k-1} \frac{e_j}{e_k^2} (m_{j+1} - m_j) \right) p_{2i-1}^{k+1} p_{2u-1}^{k+1}.$$

On the other hand, since the branches at the point $P_{0,1}^{k+1}$ are the branches of the polar of a genus one curve, using Theorem 2.2, one gets

$$I_{P_{0,1}^{k+1}}(\xi_{i,j}^{k+1}, \xi_{u,v}^{k+1}) = q_{2u-1}^{k+1} p_{2i-1}^{k+1}.$$

Summing up and using Noether's formula, one gets that

$$I(\xi_{i,j}^{k+1}, \xi_{u,v}^{k+1}) = \left(\frac{e_j}{e_k^2} + \sum_{w=1}^{k-1} \frac{e_j}{e_k^2} (m_{w+1} - m_w) \right) p_{2i-1}^{k+1} p_{2u-1}^{k+1} + q_{2u-1}^{k+1} p_{2i-1}^{k+1}.$$

Case 2. The branches are in distinct packages.

Consider $\xi_{i,j}^{l+1}$ and $\xi_{u,v}^{k+1}$ where $0 \leq l < k$, $1 \leq i \leq \left\lceil \frac{s(l+1)+1}{2} \right\rceil$, $1 \leq u \leq \left\lceil \frac{s(k+1)+1}{2} \right\rceil$, $1 \leq j \leq h_{2i}^{l+1}$ and $1 \leq v \leq h_{2u}^{k+1}$.

We have that the sum of products of the multiplicities until the point $P_{s(l), h_{s(l)}}^l$ is

$$\frac{p_{2i-1}^{l+1} p_{2u-1}^{k+1}}{e_l e_k} \left(n m_1 + \sum_{w=1}^{l-1} e_w (m_{w+1} - m_w) \right),$$

while the sum of products of the multiplicities at the remaining points is

$$\frac{p_{2i-1}^{l+1} p_{2u-1}^{k+1}}{e_l e_k} (m_{l+1} - m_l) e_l.$$

Therefore, if $e_0 = n$ and $m_0 = 0$, then

$$\begin{aligned} I(\xi_{i,j}^{l+1}, \xi_{u,v}^{k+1}) &= \frac{p_{2i-1}^{l+1} p_{2u-1}^{k+1}}{e_l e_k} \left(n m_1 + \sum_{w=1}^l e_w (m_{w+1} - m_w) \right) \\ &= \frac{p_{2i-1}^{l+1} p_{2u-1}^{k+1}}{e_l e_k} \left(\sum_{w=1}^l m_w (e_{w-1} - e_w) + m_{l+1} e_l \right). \end{aligned}$$

By construction and by an analogous computation, we may show that

$$\frac{I(\xi_{i,j}^{l+1}, f)}{m(\xi_{i,j}^{l+1})} = \frac{1}{n} \left(\sum_{w=1}^l m_w (e_{w-1} - e_w) + m_{l+1} e_l \right).$$

In this way, we see that ξ_u is precisely the u -th package in Merle's Theorem. \square

From the above theorem we get immediately the following result:

Corollary 3.2. *The number of branches of the j -th package ξ_j in Merle's decomposition of the polar of a general member of $K(n, m_1, \dots, m_r)$ is given by*

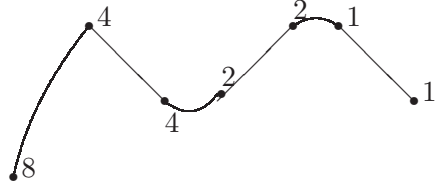
$$\sum_{k=1}^{\lfloor \frac{s(j)+1}{2} \rfloor} h_{2k}^j,$$

where the numbers that appear in the formula are obtained from the euclidean divisions described in (2.2).

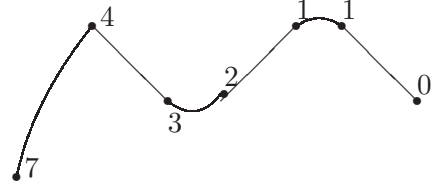
Example 3.3. *Let f be general member of $K(8, 12, 14, 15)$. The Euclidean divisions in this case are:*

$m_1 = 12$ and $n = 8$	$m_2 - m_1 = 2$ and $e_1 = 4$	$m_3 - m_2 = 1$ and $e_2 = 2$
$12 = 1(8) + 4$	$2 = 0(4) + 2$	$1 = 0(2) + 1$
$8 = 2(4)$	$4 = 2(2)$	$2 = 2(1)$

In this way, we have

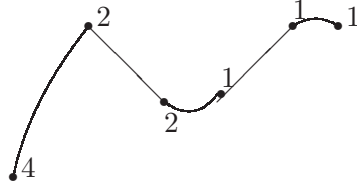


Enriques' diagram of f

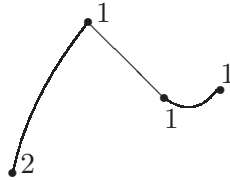


Enriques' diagram of $P(f)$

Since $h_0^3 = 0$, $h_1^3 = 1$ and $h_2^3 = 1$, according to Theorem 3.1, the third package ξ_3 of $P(f)$ has just one branch $\xi_{1,1}^3 \in K(4,6,7)$ whose Enriques' diagram is



Now, since $h_0^2 = 0$, $h_1^2 = 1$ and $h_2^2 = 1$, the second package ξ_2 of $P(f)$ has just one branch $\xi_{1,1}^2 \in K(2,3)$, whose Enriques' diagram is



Finally, the first package is ξ_1 , corresponding to the polar of a general member of $K(2,3)$, hence it has one smooth branch $\xi_{1,1}$, whose Enriques' diagram is



For the intersection multiplicities of these branches, the theorem gives us

$$I(\xi_{1,1}, \xi_{1,1}^2) = 3, I(\xi_{1,1}, \xi_{1,1}^3) = 6, I(\xi_{1,1}^2, \xi_{1,1}^3) = 13.$$

From Merle's Theorem it follows that each branch of the j -th Merle's package ξ_j of the polar of an irreducible curve has genus at least $j - 1$. On the other hand, from the proof of Theorem 3.1 one may see that the genus of each component of ξ_j is less or equal than j , when the curve is general in its equisingularity class. This generality condition is a sufficient condition to guarantee the bound j from above for the genus of the components of ξ_j , as one may see in Remark 2.1 of [HHI2].

The problem we address now is to characterize the equisingularity classes given by $K(n, m_1, \dots, m_r)$ for which the general member has its polar curve composed by branches with genus up to $r - 1$.

Corollary 3.4. *Let f be a power series corresponding to a general member of $K(n, m_1, \dots, m_r)$. The polar of f has branches of genus at most $r - 1$, if and only if $m_r = m_{r-1} + \lambda e_{r-1} - 1$, for some integer $\lambda \geq 1$.*

Proof: From Theorem 3.1, this happens if and only if the $\xi_{i,j}^r$ have genus $r - 1$. Since $\xi_{i,j}^r \in K(p_{2i-1}^r \frac{n}{e_{r-1}}, p_{2i-1}^r \frac{m_1}{e_{r-1}}, \dots, p_{2i-1}^r \frac{m_{r-1}}{e_{r-1}}, p_{2i-1}^r \frac{m_{r-1}}{e_{r-1}} + q_{2i-1}^r)$, this, in turn, happens if and only if $p_{2i-1}^r = 1$ for all $i = 1, \dots, t(r)$, where $s(r) = 2t(r)$. Now, since the p_j^r form an increasing sequence, one must have $t(r) = 1$. We have two possibilities:

1) $m_r - m_{r-1} = h_0^r e_{r-1} + n_1^r$, $e_{r-1} = h_1^r n_1^r + 1$ and $n_1^r = h_2^r \cdot 1$. Now, since $\frac{q_1^r}{p_1^r} = [h_0^r, h_1^r]$ is an integer, we must have $h_1^r = 1$. Therefore, the condition that $P(f)$ has branches of genus at most $r - 1$ is equivalent to

$$m_r - m_{r-1} = (h_0^r + 1)e_{r-1} - 1.$$

2) $m_r - m_{r-1} = h_0^r e_{r-1} + 1$ and $e_{r-1} = (h_1^r - 1) \cdot 1 + 1$. Since $\frac{q_1^r}{p_1^r} = [h_0^r, h_1^r - 1]$ is an integer, then $h_1^r = 2$. Which gives $e_{r-1} = 2$. Therefore, the condition that $P(f)$ has branches of genus at most $r - 1$ is equivalent to

$$m_r - m_{r-1} = (h_0^r + 1)e_{r-1} - 1.$$

Concluding in this way our proof. □

References

- [C1] CASAS-ALVERO, E.; *On the singularities of polar curves*. Manuscripta Math. 43, 167-190, (1983).
- [C2] CASAS-ALVERO, E.; *Infinitely near imposed singularities and singularities of polar curves*. Math. Ann. **287**, 429-454 (1990).

- [C3] CASAS-ALVERO, E.; *Singularities of Plane Curves*. Cambridge University Press. (2000).
- [HHI1] HEFEZ, A; HERNANDES, M. E. AND IGLESIAS, M. F. H.; *Plane branches with Newton nondegenerate polars*. Arxiv: 160.07522[math.AG].
- [HHI2] HEFEZ, A; HERNANDES, M. E. AND IGLESIAS, M. F. H.; *On Polars of Plane Branches*. Arxiv: 1505.08038[math.AG], to appear in the volume dedicated to Jose Seade, Trends in Mathematics, Birkhäuser.
- [M] MERLE, M.; *Invariants Polaires des Courbes Planes*. Inventiones Mathematicae, **41**, 103-111, (1977).
- [P] PHAM F.; *Deformations equisingulières des idéaux jacobiens des courbes planes*. In Proc. of Liverpool Symposium on Singularities II, Volume 209 of Lect. Notes in Math., Springer Verlag, Berlin, London, New York, 218-233, (1971).
- [T] TEISSIER B.; *Variétés polaires I: Invariants polaires des singularités d'hypersurfaces*. Inventiones Mathematicae, **41**, no. 3, 267-292, (1977).